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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1242

THEORY OF CHARACTERISTICS

By Dr. W. Tollmien

Translation of "Charakteristikentheorie."
Technische Hochschule Dresden, Archiv Nr. 44/2, Chapter II.

Washington
September 1949

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Outline:

1. Introduction

The theory of characteristics will be presented generally for quasilinear differential equations of the second order in two variables. This is necessary because of the manifold requirements to be demanded from the theory of characteristics.

The function χ of the two independent variables x, y is assumed as satisfying the quasilinear differential equation

$$A\chi_{xx} + B\chi_{xy} + C\chi_{yy} + D = 0 \tag{1}$$

This differential equation is called quasilinear since the highest derivatives, namely those of the second order, occur only linearly. The coefficients A, B, C, D are functions of $x, y, \chi, \chi_x, \chi_y$. The differential equations of gas flows set up in chapter I all belong to

*"Charakteristikentheorie." Technische Hochschule Dresden, Archiv Nr. 44/2, Chapter II.

this type. It will occasionally be useful to interpret the desired integral $\chi(x, y)$ of the differential equation (1) geometrically as integral surface in the x, y, χ - space.

The characteristics are to be introduced in three ways:

First, as loci of the possible appearance of small disturbances. For gas flows this interpretation is obvious: the characteristic base curves here are nothing but the Mach waves obtained in the known elementary manner by superposition of sound waves according to Huygens' principle as fronts of a weak disturbance wave. Following it will be defined of what type the disturbances or discontinuities are which are propagated for instance from the boundary of the region into the interior along the characteristics.

The second interpretation of the characteristics will start from the fundamental fact that the characteristics are the sole curves from which in general the integral surface $\chi(x, y)$ can be constructed.

The third introduction of the characteristics is the one used most frequently in mathematical representations. It shows in what sense the continuation of an integral surface beyond a characteristic may become indefinite. This definition of characteristics is unnecessary for our purposes. It is mentioned merely in order to ensure connection with the customary mathematical literature; however, omission of this section is not detrimental to the understanding of the rest.

The purpose of this chapter is attained with the development of two general approximation methods for the solution of the characteristic differential equation system.

2. Preliminary Statements

The characteristics lie on the integral surface $\chi(x, y)$. Their projections on the x-y plane, or, in other words, their base projections, are designated as "characteristic base curves". Let these characteristic base curves have the equation

$$\eta(x,y) = \text{Constant} \tag{2}$$

For reasons of a simpler manner of expression, the characteristic base curve

$$\eta(x,y) = 0 \tag{2a}$$

will be considered (fig. 1.) The characteristic family of curves (2) may be intersected by another family of curves.

$$\xi(x,y) = \text{Constant} \tag{3}$$

No further data are given concerning this second family of curves; for hyperbolic differential equations where two families of characteristics appear, the second family of characteristic base curves will be selected as ξ - family. This is mentioned only incidentally; for the immediately following considerations only η = Constant are assumed as characteristic base curves which are intersected by the curves ξ = Constant. Thus one has as coordinate along a characteristic curve, ξ, as transverse coordinate, η. The derivative of a function $f(x, y)$ with respect to ξ along a characteristic base curve (as which $\eta = 0$ will be selected below) will be called "interior derivative"; to attain it, nothing but the course of the function within the considered ξ - region on $\eta = 0$ is needed. Derivatives with respect to η require knowledge of the behavior of the function to be derived outside of the characteristic curve; they will be called "exterior derivatives". It is obvious that the conceptions of interior and exterior derivatives are very closely connected with the conceptions of tangential and normal derivatives.

After these preliminary statements the announced definitions of characteristics are set up.

3. Characteristics as Loci of Discontinuities of the Second Order

The integral function χ and its first derivatives are to remain continuous when the characteristic $\eta = 0$ is transversed. Discontinuities in the second derivatives - the highest ones occurring in the differential equation (1) - are to be permissible but with the restriction that at least the interior or tangential derivatives still remain continuous. The permitted discontinuities concern at most the exterior derivatives of the derivatives of the first order.

These properties are best formulated in the ξ, η - coordinate system. Obviously this is permissible since the geometrical interpretations of the conceptions used for definition of characteristics are independent of any coordinate system. χ is the height of the integral surface above the base plane, the first derivatives of χ

with respect to any two coordinates define the position[1] of the surface elements. Interior and exterior derivatives of the first derivatives of χ, therefore, measure the variation of the position of the surface elements along and across the curve $\eta = 0$.

According to the definition given above χ, χ_ξ, χ_η are, therefore, to remain continuous when $\eta = 0$ is crossed, furthermore the interior derivatives of χ_ξ and χ_η, thus $\chi_{\xi\xi}$ and $\chi_{\eta\xi}$. The single permitted discontinuity of the second order may, therefore, appear only in $\chi_{\eta\eta}$. The discontinuity the quantity $\chi_{\eta\eta}$ undergoes in passing from negative to positive will be designated by $\left[\chi_{\eta\eta}\right]$.

In order to set up the equation of $\eta = 0$ in the x, y coordinates one has to go back to the original coordinates.

$$\begin{aligned}
\chi_{xx} &= \chi_{\eta\eta}\eta_x^{\,2} + 2\chi_{\eta\xi}\eta_x\xi_x + \chi_{\xi\xi}\xi_x^{\,2} + \chi_\eta\eta_{xx} + \chi_\xi\xi_{xx}, \\
\chi_{xy} &= \chi_{\eta\eta}\eta_x\eta_y + \chi_{\eta\xi}(\eta_x\xi_y + \eta_y\xi_x) + \chi_{\xi\xi}\xi_x\xi_y + \chi_\eta\eta_{xy} + \chi_\xi\xi_{xy}, \\
\chi_{yy} &= \chi_{\eta\eta}\eta_y^{\,2} + 2\chi_{\eta\xi}\eta_y\xi_y + \chi_{\xi\xi}\xi_y^{\,2} + \chi_\eta\eta_{yy} + \chi_\xi\xi_{yy}.
\end{aligned} \tag{4}$$

Consequently the discontinuities in χ_{xx}, χ_{xy}, χ_{yy} will be

$$\begin{aligned}
\left[\chi_{xx}\right] &= \left[\chi_{\eta\eta}\right]\eta_x^{\,2}, \\
\left[\chi_{xy}\right] &= \left[\chi_{\eta\eta}\right]\eta_x\eta_y, \\
\left[\chi_{yy}\right] &= \left[\chi_{\eta\eta}\right]\eta_y^{\,2}.
\end{aligned} \tag{5}$$

[1]The position of the surface elements is defined by the direction of their normals, the direction-cosines of which in the x-y-χ system are the proportion $\chi_x : \chi_y : -1$.

If one now sets up the differential equation (1) for absolutely small positive and negative η for the same ξ and forms the difference, one obtains

$$A\left[X_{xx}\right] + B\left[X_{xy}\right] + C\left[X_{yy}\right] = 0, \tag{6}$$

or according to (5)

$$A\eta_x^{\ 2} + B\eta_x\eta_y + C\eta_y^{\ 2} = 0. \tag{7}$$

This equation will be designated as "characteristic condition".

If one expresses the characteristic base curve $\eta = 0$ in the parameter form $x = x(\xi)$; $y = y(\xi)$, one may write, since the slope is $\frac{y_\xi}{x_\xi} = -\frac{\eta_x}{\eta_y}$, for the characteristic condition (7) also:

$$A\dot{y}^2 - B\dot{y}\dot{x} + C\dot{x}^2 = 0, \tag{8}$$

where the differentiations with respect to the parameter are denoted by dots above them.

From this quadratic relation one may for hyperbolic differential equations where

$$B^2 - 4AC > 0 \tag{9}$$

set up the differential equations for two characteristic families of base curves.

Whereas the occurrence of discontinuities of the second order is restricted by the characteristic condition, the differential equation (1) does not offer a condition for the existence of discontinuities of the first order. One can see this readily if one assumes on each side of a space curve, selected as carrier of the initial conditions, two different positions of the surface elements prescribed for the integral surfaces. Discontinuities of the first order appear in gas dynamics as compression shocks.

4. Characteristic Strips as Elements of the Integral Surface

For the second interpretation of the characteristics one at first orders with regard to a curve $\eta = 0$ on a surface X the positions of the surface elements. One then speaks of a strip of the first order α_1. The surface X is not a priori assumed to be an integral surface. The position of the surface elements is suitably determined by the derivatives of X with respect to the coordinates ξ and η; thus $X_\xi = p_1$, $X_\eta = q_1$. The strip α_1 is determined in parametric representation, thus by $x(\xi)$, $y(\xi)$, $X(\xi)$, $p_1(\xi)$, $q_1(\xi)$.

The problem arises whether it is possible to express the quasilinear differential expression on the left side of the differential equation (1):

$$AX_{xx} + BX_{xy} + CX_{yy} + D \tag{10}$$

merely by the five strip quantities $x(\xi)$, $y(\xi)$, $X(\xi)$, $p_1(\xi)$, $q_1(\xi)$ and their derivatives with respect to ξ. One then says that the differential expression (10) lies in the strip α_1 or that it is an "interior differential expression" of the strip α_1.

Obviously one may then hope to satisfy the differential equation (1) along such a strip without leaving it in a transverse direction, and to build up the desired integral surface from such strips.

According to presupposition, the coefficients A, B, C, D depend only on x, y, X, p_1, q_1 which are given directly on the strip. Thus it remains only to be determined when X_{xx}, X_{xy}, X_{yy} can be expressed merely by the five strip quantities and their interior derivatives. According to the formulas (4), introducing as far as possible the strip quantities, one obtains:

$$\begin{aligned}
X_{xx} &= q_{1\eta}\eta_x^2 + 2q_{1\xi}\eta_x\xi_x + p_{1\xi}\xi_x^2 + q_1\eta_{xx} + p_1\xi_{xx} \\
X_{xy} &= q_{1\eta}\eta_x\eta_y + q_{1\xi}(\eta_x\xi_y + \eta_y\xi_x) + p_{1\xi}\xi_x\xi_y + q_1\eta_{xy} + p_1\xi_{xy} \\
X_{yy} &= q_{1\eta}\eta_y^2 + 2q_{1\xi}\eta_y\xi_y + p_{1\xi}\xi_y^2 + q_1\eta_{yy} + p_1\xi_{yy}
\end{aligned} \tag{11}$$

One sees immediately that the sole term which cannot be formed by interior differentiation in α_1 (that is with respect to ξ) always stands first on the right side of these expressions (11). Thus one obtains as sole differential contribution of (10)

$$q_{1\eta}(A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2)$$

In order to have the differential expression (10) lie completely in α_1, it is necessary and sufficient that

$$A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$$

This is again the characteristic relation (7). Thus none but the characteristic strips can be concerned in the building up of an integral surface from strips. Of course, it remains still to be shown that the building up from the characteristic strips is actually possible.

5. Indefinite Continuation of an Integral Surface Beyond a Characteristic Strip

In order to obtain the third customary definition of characteristics one starts from a strip of the first order on the integral surface X. Let the strip be determined as function of the parameter ξ by the five quantities $x, y, X, X_x = p, X_y = q$. Due to the meaning of p and q there exists between the five strip quantities x, y, X, p, and q the relation:

$$\frac{dX}{d\xi} = \frac{\partial X}{\partial x}\frac{dx}{d\xi} + \frac{\partial X}{\partial y}\frac{dy}{d\xi} = p\frac{dx}{d\xi} + q\frac{dy}{d\xi}$$

or, if one again denotes the differentiations with respect to the parameter by dots above them:

$$\dot{X} = p\dot{x} + q\dot{y} \tag{12}$$

This is the so-called "strip relation of the first order".

Let the derivatives of the second order which are not given be denoted by $r = X_{xx}$, $s = X_{xy}$, $t = X_{yy}$. The question arises whether it is always possible to determine uniquely with the aid of the differential equation (1) on the strip of the first order the derivatives of the second order r, s, and t and furthermore the derivatives of higher order so that a continuation of the integral χ by a Taylor series appears possible. A well-known theorem of Sonja Kowalewskaja deals with this analytical continuation of an integral.

For the determination of r, s, t from the five strip quantities x, y, χ, p, q the differential equation (1), for one, is at disposal, now written as follows:

$$Ar + Bs + Ct = -D \tag{13}$$

From the meaning of r, s, t one obtains the strip relations of the second order:

$$\frac{dp}{d\xi} = \frac{\partial p}{\partial x}\frac{dx}{d\xi} + \frac{\partial p}{\partial y}\frac{dy}{d\xi} = r\frac{dx}{d\xi} + s\frac{dy}{d\xi} ,$$

for which one writes briefly:

$$\dot{x}r + \dot{y}s = \dot{p} \tag{14}$$

Correspondingly, one obtains as further strip relation:

$$\dot{x}s + \dot{y}t = \dot{q} \tag{15}$$

From the three linear equations

$$\begin{aligned} Ar + Bs + Ct &= -D && (13)\\ \dot{x}r + \dot{y}s &= \dot{p} && (14)\\ \dot{x}s + \dot{y}t &= \dot{q} && (15) \end{aligned}$$

r, s, t may be determined uniquely, if the following determinant is not zero:

$$\begin{vmatrix} A & B & C \\ \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \end{vmatrix} = A\dot{y}^2 - B\dot{x}\dot{y} + C\dot{x}^2$$

It is interesting to note that furthermore, in order to make the higher derivatives uniquely determinable, non-vanishing of the same determinant is necessary and sufficient[2].

If the determinant vanishes which thus coincides with the characteristic condition derived before (8), r, s, t are no longer uniquely determinable, but - if at all - only with exclusion of additive solutions of the homogeneous equation system pertaining to (13), (14), (15). This new criterion of the characteristic condition is, of course, very closely connected with the property of the characteristics (used in section 3) of being geometrical locus for the discontinuities of the second order of the integral surface.

If solutions are to exist at all when the determinant of the equation system (13), (14), (15) vanishes, additional conditions must exist between the coefficients of the left and right sides. Since the rank of the matrix

$$\begin{pmatrix} A & B & C & -D \\ \dot{x} & \dot{y} & 0 & \dot{p} \\ 0 & \dot{x} & \dot{y} & \dot{q} \end{pmatrix}$$

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[2]In order to find for instance r_x, s_x, t_x, one may, by differentiation of (13) with respect to x, set up the equation

$$Ar_x + Bs_x + Ct_x = -\left[A_x r + B_x s + C_x t + D_x\right]$$

as well as the two strip relations of the third order

$$\dot{x}r_x + \dot{y}s_x = \dot{r},$$

$$\dot{x}s_x + \dot{y}t_x = \dot{s}.$$

For r, s, t assumed as already determined the condition mentioned results for the unique determination of r_x, s_x, t_x.

must be smaller than 3, it is necessary that the following determinant be zero:

$$\begin{vmatrix} A & C & -D \\ \dot{x} & 0 & \dot{p} \\ 0 & \dot{y} & \dot{q} \end{vmatrix} = A\dot{p}\dot{y} + C\dot{q}\dot{x} + D\dot{x}\dot{y} = 0 \tag{16}$$

According to a determinant theorem found by Kronecker[3] this condition is sufficient. This will not be discussed in more detail since another derivation will be given directly for this second equation (16) of the characteristic strip which is added to the characteristic condition (8). As third equation for the characteristic strip one has the strip relation of the first order (12) already mentioned:

$$\dot{\chi} = p\dot{x} + q\dot{y}$$

6. The Characteristic Differential Equation System

The characteristic condition

$$A\dot{y}^2 - B\dot{x}\dot{y} + C\dot{x}^2 = 0 \tag{8}$$

introduced in three different ways yields the differential equation for the characteristic base curves. In order to prepare, according to the deliberations in section 4, for the building up of the integral surface from strips of the first order on the integral surface $\chi(x, y)$, one has, furthermore, to set up differential equations for the height and the position of the integral-surface elements which one now visualizes as described by $\chi = \chi(\xi)$, $\chi_x = p(\xi)$, $\chi_y = q(\xi)$. Due to the meaning of p and q as derivatives of χ with respect to x and y, the relation

$$\frac{d\chi}{d\xi} = \frac{\partial\chi}{\partial x}\frac{dx}{d\xi} + \frac{\partial\chi}{\partial y}\frac{dy}{d\xi}$$

[3]See for instance M. Bôcher: Einführung in die höhere Alegbra, (Introduction into the higher algebra), chapter V, 19, first theorem.

must be valid, thus, again denoting differentiations with respect to the parameter by dots above them,

$$\dot{\chi} = p\dot{x} + q\dot{y} \tag{17}$$

This is the so-called "strip relation of the first order".

The given differential equation (1) results after multiplication by $\dot{x}\dot{y}$ and the designations $r = \chi_{xx}$, $s = \chi_{xy}$, $t = \chi_{yy}$:

$$Ar\dot{x}\dot{y} + Bs\dot{x}\dot{y} + Ct\dot{x}\dot{y} + D\dot{x}\dot{y} - 0 \tag{18}$$

According to the meaning of p, q, r, s, t one obtains the strip relation of the second order:

$$\left.\begin{aligned} \dot{p} &= r\dot{x} + s\dot{y}, \\ \dot{q} &= s\dot{x} + t\dot{y}. \end{aligned}\right\} \tag{19}$$

If one inserts the expression for $r\dot{x}$ and $t\dot{y}$ from these equations into (18), one obtains:

$$(Bs\dot{x}\dot{y} - As\dot{y}^2 - Cs\dot{x}^2) + A\dot{p}\dot{y} + C\dot{q}\dot{x} + D\dot{x}\dot{y} = 0 \tag{20}$$

The bracket vanishes according to the characteristic condition (8). Thus one obtains as second characteristic differential equation

$$A\dot{p}\dot{y} + C\dot{q}\dot{x} + D\dot{x}\dot{y} = 0 \tag{21}$$

Finally, the third characteristic differential equation is given by the strip relation of the first order (17):

$$\dot{\chi} = p\dot{x} + q\dot{y} \tag{17}$$

One therefore has a characteristic differential equation system altogether for the five strip quantities $x(\xi)$, $y(\xi)$, $X(\xi)$, $p(\xi)$, $q(\xi)$:

$$A\dot{y}^2 - B\dot{x}\dot{y} + C\dot{x}^2 = 0, \tag{22a}$$

$$A\dot{p}\dot{y} + C\dot{q}\dot{x} + D\dot{x}\dot{y} = 0, \tag{22b}$$

$$\dot{X} = p\dot{x} + q\dot{y}. \tag{22c}$$

According to presupposition the differential equation (1) is hyperbolic, thus

$$B^2 - 4AC > 0, \tag{9}$$

so that the first characteristic condition (8) has two different real roots for y:

$$\dot{y} = \lambda'\dot{x}, \; \dot{y} = \lambda''\dot{x}. \tag{23}$$

One notes for later that $\lambda'\lambda'' = \frac{C}{A}$. The parameter of the characteristics of the first family is called ξ, as before, the one on the second family is called η. For η = Constant one than obtains just the characteristics of the first family, for ξ = Constant those of the second family (see fig. 1). If one substitutes for instance $\dot{y} = \lambda'\dot{x}$ into the second characteristic differential equation (21), one obtains

$$\dot{x}(D\lambda'\dot{x} + A\lambda'\dot{p} + C\dot{q}) = 0,$$

or

$$A\lambda'\dot{p} + C\dot{q} + D\dot{y} = 0. \tag{24}$$

The third equation (17) or (22c), respectively, of the characteristic differential equation system may remain unchanged.

Thus one obtains for the first family of characteristics the three differential equations

$$y_\xi - \lambda' x_\xi = 0, \tag{25a}$$

$$A\lambda' p_\xi + Cq_\xi + Dy_\xi = 0, \tag{25b}$$

$$X_\xi - px_\xi - qy_\xi = 0. \tag{25c}$$

For the second family of characteristics one obtains, correspondingly:

$$y_\eta - \lambda'' x_\eta = 0, \tag{26a}$$

$$A\lambda'' p_\eta + Cq_\eta + Dy_\eta = 0, \tag{26b}$$

$$X_\eta - px_\eta - qy_\eta = 0. \tag{26c}$$

If one interprets the differential equations (25) and (26), respectively, as ordinary differential equations for $x(\xi)$, $y(\xi)$, $X(\xi)$, $p(\xi)$, $q(\xi)$ and $x(\eta)$, $y(\eta)$, $X(\eta)$, $p(\eta)$, $q(\eta)$, respectively, one has two indeterminate systems so that from this standpoint one cannot arrive at an integration theory of the partial differential equation (1).

However, if one regards the characteristics as Gauss parameter curves on the integral surface, so that the latter is described by $X(\xi,\eta)$, $x(\xi,\eta)$, $y(\xi,\eta)$ one may now interpret the six differential equations (25), (26) as partial differential equation system for the five quantities $x(\xi,\eta)$, $y(\xi,\eta)$, $X(\xi,\eta)$, $p(\xi,\eta)$, $q(\xi,\eta)$ which seems overdetermined.

It is also of importance for the practical calculation that the last equation (26c) of the differential equation system is automatically fulfilled when the first five are satisfied. It is, therefore, not a case of overdetermination.

In order to prove this important fact, one multiplies (25b) by y_η, (26b) by y_ξ and subtracts:

$$A(\lambda' p_\xi y_\eta - \lambda'' p_\eta y_\xi) + C(q_\xi y_\eta - q_\eta y_\xi) = 0.$$

One divides this equation by $\lambda'\lambda'' = \frac{C}{A}$:

$$p_\xi \frac{y_\eta}{\lambda''} - p_\eta \frac{y_\xi}{\lambda'} + q_\xi y_\eta - q_\eta y_\xi = 0$$

Hence one obtains, using (25a) and (26a):

$$p_\xi x_\eta - p_\eta x_\xi + q_\xi y_\eta - q_\eta y_\xi = 0,$$

or

$$p_\eta x_\xi + q_\eta y_\xi = p_\xi x_\eta + q_\xi y_\eta. \tag{27}$$

If one now differentiates (25c) with respect to η:

$$\chi_{\xi\eta} - p_\eta x_\xi - q_\eta y_\xi - p x_{\xi\eta} - q y_{\xi\eta} = 0$$

and substitutes the relation (27) found above:

$$\chi_{\xi\eta} - p_\xi x_\eta - q_\xi y_\eta - p x_{\xi\eta} - q y_{\xi\eta} = 0,$$

one recognizes that

$$\chi_\eta - p x_\eta - q y_\eta$$

is a function of η solely which will be called $h(\eta)$. If the strip relation (26c) is satisfied on a boundary curve cut from the curves η = Constant (see fig. 2) that is, the first family of characteristics, $h(\eta)$ vanishes; consequently, the strip relation (26c) is satisfied also in the entire interior[4]. The condition used just now, that the strip relations are satisfied at the boundaries of the region, must, of course, be duly taken into account in prescribing initial or boundary conditions.

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[4]In this conclusion it is assumed that characteristics of the first family which are in the interior of the region somewhere meet the boundary. Characteristics the entire course of which lies in the interior of a region would represent a rare exception which may be disregarded.

The existence of the solutions of the differential equation system (25), (26) was first established for a pure initial-value problem of (1) by K. Friedrichs and H. Lewy, for the mixed problems with initial and boundary conditions which are of particular importance for us by F. Frankl and R. Aleksejeva. These existence theorems are not so interesting at the moment since we may refer to the physical evidence of the possibility of solution. Later on, it is true, we shall have to deal with the restricting presuppositions made by F. Frankl and R. Aleksejeva for their existence proof, since they seem to be connected with certain occurrences of physical interest.

7. Construction of an Approximate Solution of the Characteristic Differential Equation System According to the Lattice Point and Field Method

One assumes the six characteristic differential equations of the form (25), (26) written in differentials:

$$dy - \lambda' dx = 0, \tag{28a}$$

$$A\lambda' dp + C dq + D dy = 0, \tag{28b}$$

$$d\chi - p dx - q dy = 0; \tag{28c}$$

$$dy - \lambda'' dx = 0, \tag{29a}$$

$$A\lambda'' dp + C dq + D dy = 0, \tag{29b}$$

$$d\chi - p dx - q dy = 0. \tag{29c}$$

The three equations (28) refer to the first family of characteristics (slope of the base curves λ'), the three equations (29) to the second family of characteristics (slope of the base curves λ''). One of the two equations (28c) or (29c) may be omitted according to the expositions of the last paragraph.

This presentation of the characteristic differential equations immediately suggests an approximation method which essentially consists in the replacement of the differentials by finite differences.

One visualizes the x, y - region covered by a net of the two families of characteristic base curves. In figure 3 the characteristic base curves of the first family with the slope λ' are drawn in solid lines, those of the second family with the slope λ'' in dashed ones. The points of intersection form a point lattice. According to the first method, the "lattice point method", the position of these lattice points and the values of χ, p, q at the lattice points are to be determined approximately.

A rule for continuation is given by assuming that the procedure is performed to include the lattice points 1, 2, whereas the continuation is to take place toward the lattice point 3. The indices are apportioned according to the numbers of the lattice points. By replacing, as announced, the differentials in (28a) and (29a) by differences, one obtains for the approximate determination of x, y, that is the position of the lattice point 3, the two equations

$$y_3 - y_1 = \lambda_1'(x_3 - x_1), \tag{30a}$$

$$y_3 - y_2 = \lambda_2''(x_3 - x_2). \tag{30b}$$

The approximate values of p_3, q_3 at the lattice point 3 are, by approximation of the differential equations (28b), (29b), determined correspondingly from the following two equations:

$$A_1\lambda_1'(p_3 - p_1) + C_1(q_3 - q_1) + D_1(y_3 - y_1) = 0, \tag{31a}$$

$$A_2\lambda_2''(p_3 - p_2) + C_2(q_3 - q_2) + D_2(y_3 - y_2) = 0. \tag{31b}$$

Finally one may determine χ_1 according to (28c) or (29c) approximately from

$$\chi_3 - \chi_1 - p_1(x_3 - x_1) - q_1(y_3 - y_1) = 0, \tag{32a}$$

or from

$$\chi_3 - \chi_2 - p_2(x_3 - x_2) - q_2(y_3 - y_2) = 0 \tag{32b}$$

The second method, the "field method", offers certain advantages as to the representation of the results while the expenditure of calculation is the same as in the lattice point method. In the field method one starts from the concept that the characteristic base curves divide the x,y - region into fields. For every one of these fields an approximate value for each p and q is to be determined. The distribution of χ will then be represented by lines of equally large χ (contour lines of the integral surface). According to this interpretation, a calculation scheme is formed by the characteristic base curves; in each compartment or field two figures, namely the approximate values of p and q, are written.

The rule for continuation for the field method is developed from that for the lattice point method. One assumes the field method to have proceeded so far that the desired approximation values in the fields I and II (fig. 4) are already known whereas they are just about to be determined for the adjoining field III. Correspondingly, the coordinated lattice point method is assumed to have proceeded to include the points 1, 2, 3, whereas it is just about to be applied to the lattice points 4 and 5. The data about the approximation values obtained from the characteristic differential equations (28b), (29b) are of foremost importance; they read, in appropriate sequence:

$$A_1\lambda_1'(p_4 - p_1) + C_1(q_4 - q_1) + D_1(y_4 - y_1) = 0, \tag{33a}$$

$$A_2\lambda_2'(p_5 - p_2) + C_2(q_5 - q_2) + D_2(y_5 - y_2) = 0; \tag{33b}$$

$$A_2\lambda_2''(p_4 - p_2) + C_2(q_4 - q_2) + D_2(y_4 - y_2) = 0, \tag{34a}$$

$$A_3\lambda_3''(p_5 - p_3) + C_3(q_5 - q_3) + D_3(y_5 - y_3) = 0. \tag{34b}$$

By addition of the two equations (33a) and (33b) which have been divided by two there follows after appropriate rearrangement

$$\left.\begin{aligned}
&\frac{A_1 + A_2}{2}\,\frac{\lambda_1' + \lambda_2'}{2}\left(\frac{p_4 + p_5}{2} - \frac{p_1 + p_2}{2}\right) + \frac{C_1 + C_2}{2}\left(\frac{q_4 + q_5}{2} - \frac{q_1 + q_2}{2}\right)\\
&+ \frac{D_1 + D_2}{2}\left(\frac{y_4 + y_5}{2} - \frac{y_1 + y_2}{2}\right)\\
&= \frac{A_2 - A_1}{2}\,\frac{\lambda_1' + \lambda_2'}{2}\left(\frac{p_4 - p_5}{2} - \frac{p_1 - p_2}{2}\right)\\
&+ \frac{A_1 + A_2}{2}\,\frac{\lambda_2' - \lambda_1'}{2}\left(\frac{p_4 - p_5}{2} - \frac{p_1 - p_2}{2}\right)\\
&+ \frac{A_2 - A_1}{2}\,\frac{\lambda_2' - \lambda_1'}{2}\left(\frac{p_1 + p_2}{2} - \frac{p_4 + p_5}{2}\right) + \frac{C_2 - C_7}{2}\left(\frac{q_4 - q_5}{2} - \frac{q_5 - q_2}{2}\right)\\
&+ \frac{D_2 - D_7}{2}\left(\frac{y_4 - y_4}{2} - \frac{y_5 - y_2}{2}\right).
\end{aligned}\right\} \tag{35}$$

If - in the sense of the approximation used - one now neglects terms that are of second or higher order in the differences, the entire right side of (35) is eliminated and one obtains

$$\left.\begin{aligned}&\frac{A_1 + A_2}{2}\,\frac{\lambda_1' + \lambda_2'}{2}\left(\frac{p_4 + p_5}{2} - \frac{p_1 + p_2}{2}\right) + \frac{C_1 + C_1}{2}\left(\frac{q_4 + q_5}{2} - \frac{q_1 + q_2}{2}\right)\\ &+ \frac{D_1 + D_2}{2}\left(\frac{y_4 + y_5}{2} - \frac{y_1 + y_2}{2}\right) = 0.\end{aligned}\right\} \tag{36}$$

Now one identifies certain mean values of the desired quantities in the lattice points with the corresponding quantities in the fields to the boundary of which the respective lattice points pertain, namely[5]

$$\left.\begin{aligned}&\frac{x_1 + x_2}{2} = x_I, \quad \frac{x_2 + x_3}{2} = x_{II}, \quad \frac{x_4 + x_5}{2} = x_{III};\\ &\frac{y_1 + y_2}{2} = y_I, \quad \frac{y_2 + y_3}{2} = y_{II}, \quad \frac{y_4 + y_5}{2} = y_{III};\\ &\frac{\chi_1 + \chi_2}{2} = \chi_I, \quad \frac{\chi_2 + \chi_3}{2} = \chi_{II}, \quad \frac{\chi_4 + \chi_5}{2} = \chi_{III};\\ &\frac{p_1 + p_2}{2} = p_I, \quad \frac{p_2 + p_3}{2} = p_{II}, \quad \frac{p_4 + p_5}{2} = p_{III};\\ &\frac{q_1 + q_2}{2} = q_I, \quad \frac{q_2 + q_3}{2} = q_{II}, \quad \frac{q_4 + q_5}{2} = q_{III}.\end{aligned}\right\} \tag{37}$$

Finally $\frac{A_1 + A_2}{2}, \frac{\lambda_1' + \lambda_2'}{2}, \frac{C_1 + C_2}{2}, \frac{D_1 + D_2}{2}$ is transformed according

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[5]The point with the coordinates x_I, y_I represents in a certain sense the center of the field I, namely the bisecting point of the diagonal 1 ... 2. The definition of the field center has to be selected for the reason that of the unknown field III only the one diagonal 4 ... 5, with a course corresponding to that of 1 ... 2, is already known.

to identities of the following type:

$$\frac{A_1 + A_2}{2} = A(x_I, y_I, \chi_I, p_I, \dot{q}_I) + \frac{A(x_1, y_1, \chi_1, p_1, q_1) - A(x_I, y_I, \chi_I, p_I, q_I)}{2} + \frac{A(x_2, y_2, \chi_2, p_2, q_2) - A(x_I, y_I, \chi_I, p_I, q_I)}{2}$$

for which one writes abbreviatedly:

$$\frac{A_1 + A_2}{2} = A_I + \frac{A_1 - A_I}{2} + \frac{A_2 - A_I}{2}. \qquad (38)$$

If one substitutes this expression and the corresponding ones into (36) and again neglects terms which are of second or higher order in the differences, one obtains as a formula of the field method:

$$A_I\lambda_I'(p_{III} - p_I) + C_I(q_{III} - q_I) + D_I(y_{III} - y_I) = 0. \qquad (39)$$

The corresponding formula which can be derived from the equations (34a) and (34b) reads:

$$A_{II}\lambda_{II}''(p_{III} - p_{II}) + C_{II}(q_{III} - q_{II}) + D_{II}(y_{III} - y_{II}) = 0 \qquad (40)$$

It must especially be noted that in (39) the progressing from field I to field III takes place by crossing a characteristic base curve of the second kind whereas in (39) the slope λ' of the characteristic base curve of the first kind appears. A corresponding statement may be made regarding equation (40) which regulates the progressing from field II to field III by crossing a characteristic base curve of the first kind. For the rest, however, the equations are no more complicated than for the lattice point method.

The closing of the new field III by characteristic base curves connecting the lattice points 4 and 5 with point 6 would be caused according to the lattice point method by

$$\left.\begin{aligned} y_6 - y_4 &= \lambda_4'(x_6 - x_4) \\ y_6 - y_5 &= \lambda_5''(x_6 - x_5) \end{aligned}\right\} \tag{41}$$

If one replaces in the field method λ_4' by λ_{III}', λ_5'' by λ_{III}'':

$$\left.\begin{aligned} y_6 - y_4 &= \lambda_{III}'(x_6 - x_4) \\ y_6 - y_5 &= \lambda_{III}''(x_6 - x_5), \end{aligned}\right\} \tag{42}$$

one has altered the equations (41) only by quantities which are of second order in the differences, so that the equations (42) can be used as further equations of the field method.

The contour lines of the integral surface, that is, the lines of equally large χ, are constructed simply according to the relation

$$d\chi = p\,dx + q\,dy = 0 \tag{43}$$

as lines of the slope

$$\frac{dy}{dx} = -\frac{p}{q} \tag{44}$$

starting from an initial distribution of χ. In every field p and q remain unchanged according to the field method so that one obtains an approximation of the contour lines of the integral surface by series of lines.

The equations (39), (40), (42), and (44) determine the field method. They are no more cumbersome than the equations of the lattice point method.

So far the progressing had been represented only in the interior of the region. How the two methods have to be altered for the boundary will be indicated by an example.

Let the lattice point method be performed including the points 1, 2, 3 and be just about to be extended to the next lattice point 4 (point of intersection of a characteristic base curve of the second kind with the heavily drawn boundary, figure 5). According to (29a) one then obtains approximately

$$y_4 - y_3 = \lambda_3''(x_4 - x_3).$$

The second equation for determination of the coordinates x_4, y_4 of the lattice point 4 is yielded by the equation of the boundary curve. According to (29b) one obtains further approximately:

$$A_3\lambda_3''(p_4 - p_3) + C_3(q_4 - q_3) + D_3(y_4 - y_3) = 0.$$

The second equation for the determination of p_4, q_4 must be given by the boundary condition. χ_4 may then be approximately determined, according to (29c), from

$$\chi_4 - \chi_3 - p_3(x_4 - x_3) - q_3(y_4 - y_3) = 0.$$

According to the field method one lays a characteristic base curve of the second kind through the center of the field I (bisecting point of the diagonal 1 ... 2) to the point of intersection with the boundary. This point of intersection has been marked in figure 5 by the field number II. Within the accuracy of the field method this characteristic base curve runs parallel to 2 ... 3. Then one obtains according to (29b):

$$A_I\lambda_I''(p_{II} - p_I) + C_I(q_{II} - q_I) + D_I(y_{II} - y_I) = 0.$$

The second relation necessary for determination of p_{II}, q_{II} must be yielded by the boundary condition. The closing of the field II by a characteristic base curve of the second kind and the plotting of the curves of equal χ - values takes place in the customary manner.

The methods described yield only a first approximation. For linear problems improved approximations are often obtained by a refined approximation of the differentials in the characteristic differential equation system (28), (29) with the aid of higher differences. For nonlinear problems such an attempt does not seem very promising. Iterative methods, on the other hand, will probably lead, with tolerable expenditure of calculation, to improved approximations and finally also to an estimate of errors.

8. Literature

1. The development of the theory of characteristics given in sections 1 to 6 is governed entirely by the needs of the practice and thus deviates from the usual text book representations. However, similar interpretations may be found represented particularly in the book by R. Courant and D. Hilbert entitled "Methoden der Mathematischen Physik II" (Methods of Mathematical Physics II), Berlin 1937. Hence, for instance the concepts of interior and exterior differentiation have been taken over.

2. The existence proof for the pure initial problem has been given by K. Friedrichs and H. Lewy in Mathematische Annalen, Bd. 99, p. 200, 1928. Accounts of this existence proof may be found, for instance, in J. Hadamard's "Lecons sur le probléme de Cauchy," (Lessons regarding the problem of Cauchy) p. 487, Paris 1932 and in Courant - Hilbert l.c., p. 326. For the mixed problems occurring in the applications in gas dynamics the existence proof has been given along the same lines by F. Frankl and R. Aleksejeva in a report entitled "Two boundary-value problems from the theory of the hyperbolic partial differential equations of the second order with application to gas flows at supersonic velocity," (Russian), Matematiceski Sbornik, issue 1934, p. 483.

3. The distinction between lattice point and field method has been newly introduced here. A report on the lattice point method which was developed particularly by I. Massau, Gent 1900 to 1903, is to be found in the encyclopedia article by C. Runge and Dr. A. Willers II C 2 on numerical and graphic integration, p. 160. The field method had so far been developed only for a special case by L. Prandtl and A. Busemann, "Näherungsverfahren zur zeichnerischen Ermittelung von ebenen Strömungen mit Überschallgeschwindigkeit," Stodolafestschrift Zürich 1929 (Approximation method for graphic determination of two-dimensional flows of supersonic velocity, Stodola anniversary publication Zürich 1929). An essential simplification in this special case is based on the fact that the differential-equation coefficients A, B, C depend only on p and q whereas D altogether vanishes. For the general case treated in this report there had to be found, in contrast, above all a significant definition of the field center.

Appendix

Simplified Derivation of the Field Method

Whereas the field method was developed from the lattice point method in section 7 of chapter II, it is to be derived here directly from the characteristic differential equations.

According to the basic concept of the field method the x, y - region is divided into fields in the manner of a calculation scheme by a net of characteristic base curves; the approximate values of p and q are written in these fields. The method is assumed to have proceeded to a point where the approximate values in the fields I and II (see fig. 6) are already known whereas they are just about to be determined for the adjoining field III. The desired rule for continuation will be obtained by means of an appropriate definition of the field centers.

Since of the unknown field III 4 ... 5 is known as the sole diagonal, the bisecting point of this diagonal is selected as field center. The field centers of the fields I and II are then defined as bisecting points of the correspondingly situated diagonals 1 ... 2 and 2 ... 3. One assumes the field numbers I, II, III written at these field centers.

The corner points 1 ... 4 are, within the scope of our approximation, connected by a straight line. Since this line is closing the field I, its slope is selected to equal that of a characteristic base curve corresponding to the approximation values prevailing in the field I, thus:

$$y_4 - y_1 = \lambda_I'(x_4 - x_1). \tag{45a}$$

Similarly:

$$y_5 - y_2 = \lambda_{II}'(x_5 - x_2). \tag{45b}$$

According to the definition of the field centers there is

$$\left.\begin{aligned} &x_I = \frac{x_1 + x_2}{2},\ y_I = \frac{y_1 + y_2}{2};\ x_{II} = \frac{x_2 + x_3}{2},\ y_{II} = \frac{y_2 + y_3}{2}; \\ &x_{III} = \frac{x_4 + x_5}{2},\ y_{III} = \frac{y_4 + y_5}{2} \end{aligned}\right\} \tag{46}$$

Adding (45a) and (45b) and dividing by 2, one obtains

$$\left.\begin{aligned} \frac{y_4 + y_5}{2} - \frac{y_1 + y_2}{2} &= \lambda_I' \frac{x_4 - x_1}{2} + \lambda_{II}' \frac{x_5 - x_2}{2} \\ &= \lambda_I' \left(\frac{x_4 + x_5}{2} - \frac{x_1 + x_2}{2} \right) + \frac{(\lambda_{II}' - \lambda_I')(x_5 - x_2)}{2}. \end{aligned}\right\} \tag{47}$$

Neglecting terms which are of the second order in the differences one obtains according to (46) and (47):

$$y_{III} - y_I = \lambda_I'(x_{III} - x_I). \tag{48a}$$

Correspondingly, it can be shown that

$$y_{III} - y_{II} = \lambda_{II}''(x_{III} - x_{II}). \tag{48b}$$

Therewith it is proved that the lines connecting the field centers in first approximation are characteristic base curves.

For the transition from I to III or II to III, respectively, it is now permissible to set up the characteristic differential equation (28b) or (29b), respectively, with the differentials replaced by differences. One obtains as rule for continuation for the determination of p_{III} and q_{III}:

$$\left.\begin{aligned} A_I\lambda_I'(p_{III} - p_I) + C_I(q_{III} - q_I) + D_I(y_{III} - y_I) &= 0, \\ A_{II}\lambda_{II}'(p_{III} - p_{II}) + C_{II}(q_{III} - q_{II}) + D_{II}(y_{III} - y_{II}) &= 0. \end{aligned}\right\} \tag{49}$$

Field III is then approximately bounded by characteristic base curves:

$$\left.\begin{aligned} y_6 - y_4 &= \lambda_{III}'(x_6 - x_4), \\ y_6 - y_5 &= \lambda_{III}''(x_6 - x_5), \end{aligned}\right\}$$

if 6 is the index of the unknown corner point of field III. The corresponding rule had been assumed before for fields I and II.

The construction of the contour lines (χ = Constant) of the integral surface has already been discussed on page 20 (equations (43) and (44)), the modifications of the method becoming necessary at the boundary on page 21. The representation of these facts is not affected by the new viewpoint.

Translation by Mary L. Mahler,
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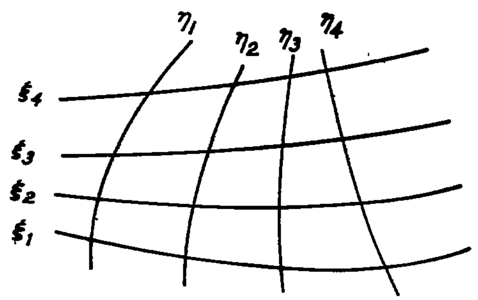


Figure 1

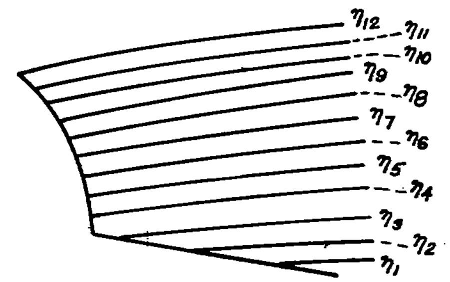


Figure 2

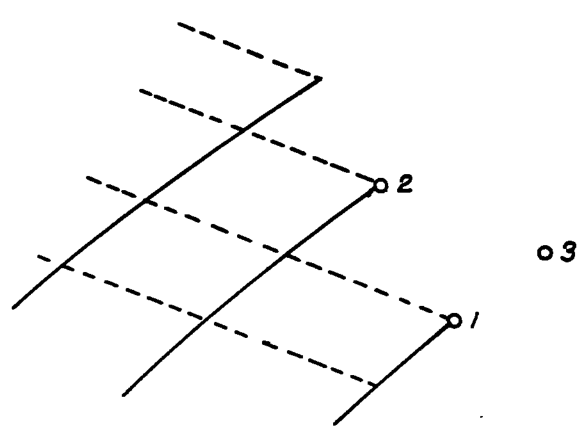
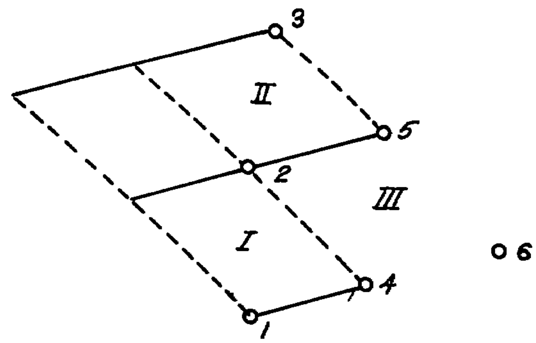


Figure 3



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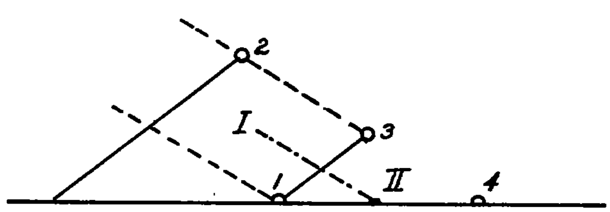

Figure 5

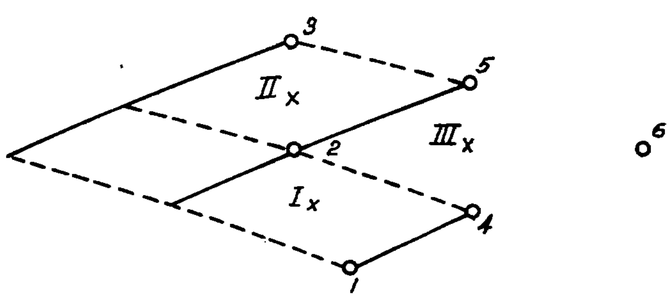

Figure 6

NACA - Langley Field, Va.